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The Quantum Pentacle

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Quantum set theory permits the formulation of a quantum simplicial topology suitable for a quantum theory of time space and gravity without prior time space structure. The quantum simplex differs strikingly from the classical: It is isotropic ("points in all directions") and all quantum simplexes of the same signature are congruent. Quantum simplexes and complexes are described by S numbers, elements of the Clifford algebra of quantum sets. The isotropy groups of noncontiguous simplexes commute, like local invariance groups in a gauge-invariant theory.

1. INTRODUCTION

We have previously explored the possibility that the universe, which we assume is a discrete network, is similar to a Penrose spin network or a Feynman diagram. Now we suggest instead that the universe is a quantum simplicial complex like a Regge skeleton. This way we can, if need be, put in *a posteriori* what we have not yet been able to get out *a priori*: the four dimensions of time space. We find that the topology of simplicial complexes may be expressed classically in the language of Grassmann algebra, and thus mates well with our quantum set theory, which is also a Grassmann algebra. Quantum simplicial topology turns out to have an important nonclassical continuous symmetry: quantum simplexes "point in all directions." To borrow a metaphor from the old vector theory of the atom, the quantum simplex spins very rapidly about any axis. Like Penrose (in his theory of spin networks) and Weizsaecker (in his Ur theory), we find that the roundness and Lorentz invariance of local time space arise as a quantum effect, like the isotropy of an electronic s state.

Ultimately, the Einstein time space must arise as a reduced coarse-grained low-temperature average. A plexic theory must be constructed without benefit of that time space, in intrinsic topological terms, like Feynman diagrams with time space removed, or Penrose–Yutzis spin networks with dynamics added, or Regge skeletons without their Minkowskian bone marrow. For this construction we use quantum algebraic topological elements we call *plexors* (Finkelstein, 1972) in general. A plexor is a tensor with an algebraic topology given on its indices. In this application the numerical value of a plexor component is the quantum amplitude for its topology. The archetype of all plexors is the following $)U)$, built out of the "small" unitary operator or "plexon" $)u)$ representing the passage of one unit of time in (say) Heisenberg's quantum theory:

$$)U) =)u))u) \ldots)u)$$

This plexor $)U)$ has the (tensor) product of n plexons $)u)$ for its tensor; and for its topology, n arcs in series, representing the passage of n units of time. The usual Feynman amplitude for this process is the diagonal part of this plexor, and is one stage in the contraction of this tensor product to the usual inner product of the n factors $)u)$.

2. BUBBLES AND PLEXONS

In quantum field theory we begin to construct a dynamical theory by naming a spacelike surface $t = t(x, y, z)$ in time space, representing an instant of time of the universe (a nonoperational concept, it would seem). Tomonaga and Schwinger generate the dynamical evolution of the system by infinitesimal stages, in each of which the instant $t(x, y, z)$ is moved forward by an infinitesimal time $dt(x, y, z)$ within an infinitesimal volume of space $dx\,dy\,dz$. Often we call the small time space volume swept out by such an evolution a "bubble"; and the functional derivative $d/dt(x, y, z)$ that appears in the Tomonaga–Schwinger Schroedinger equation, a "bubble derivative." These bubbles are filled with operators, the Hamiltonian density of the field theory.

Continuum theories suppose that such a bubble can be as small as we please. We suppose that this is no more true of these bubbles than of helium bubbles, and that there is a finite (nonzero) elementary process or *plexon*. There are many plexons in the bubbles of quantum electrodynamics, enough to give the illusion of a continuum.

The simplest volume is a simplex, and we may at first think of a plexon as an elementary simplex, if we wish to have some picture in mind. Our final version of the plexon, will indeed be a kind of simplex.

In accord with the usual quantum theory of aggregates or sets, the dynamical process is the product of its elementary constituents, or plexons and the occupation number N of these plexons composes additively when processes are multiplied. We use this additivity to guess the correspondence limit of N. The salient additive invariant of the continuum theory is the four-dimensional hypervolume V of a process (more accurately, of the bubble that supports the process), and so we suppose a fundamental proportionality

$$V=\tau^4 N$$

between 4-volume and plexon number N, the coefficient of proportionality being the fourth power of a fundamental time or chronon τ. The constant τ^4 is a fundamental quantum hypervolume or "choron."

To be sure, the action is also an additive invariant of a process, and so comes to mind as an alternative candidate for the meaning of N. But we know from existing quantum theory that action arises as the complex phase of the amplitude of the process in a special basis.

In a formal development N appears as the grade of a Grassman algebra operator representing maximal information about the quantum process. Let us consider the construction of this algebra more closely.

3. QUANTUM TOPOLOGY

If the time space of the world is quantum then it is quite likely that its topology and set theory are also. A quantum set theory has been brought to a primitive stage of development (Finkelstein, 1982) which we encapsulate here and then apply to quantum topology.

3.1. Quantum Set Theory. Quantum set theory deals with a quantum object, the quantum set, whose singlets (pure states) include all the finitary sets of classical set theory (those generated from the null set by a finite number of bracket operations and unions). The Hilbert space of the quantum set we call S. S is also a Clifford algebra of infinite dimension generated by its unity 1 (representing the null set), a monadic linear operator Br (representing bracketing), multiplication (representing Boolean addition), and addition and subtraction (representing coherent quantum superposition). This Clifford algebra has an adjoint operation * which we assume commutes with Br: $(\mathrm{Br}(s))^* = \mathrm{Br}(s^*)$. The 0 of this algebra represents the undefined; the unit 1 represents the null set. The algebra S has both a Clifford product $s's''$ and a Grassman product $s' \wedge s''$ (representing disjoint union). We call the members of S, S numbers. The center of S, its coefficient ring, is the ring Int of the integers.

S No	Grade	Dim.	Name	Diagram
1	0	−1	Null set	
a	1	0	Monad	
ab	2	1	Dyad	
abc	3	2	Triad	
abcd	4	3	Tetrad	
abcde	5	4	Pentad	

Fig. 1. The simplexes of dimension -1 to 4. The n-dimensional simplex is represented in projection most symmetrically by locating $n+1$ points spaced at equal angles on a unit circle and joining them in all possible ways by simplexes of lower dimension. Here are shown the (invisible) null simplex, the point, the line, the triangle, the tetrahedron, and the pentahedron, whose projection is the ancient pentacle.

We now apply this quantum set theory to quantum topology.

3.2. The Classical Simplex as Grassmann Product. We work here with the simplest and most algebraic topology, that of simplicial complexes. We propose that the plexon is a four-dimensional quantum simplex or pentahedron (Figure 1), suitably quantized.

An n-dimensional simplex or n-simplex may be considered as a formal product of its $n+1$ vertex points (Chevalley, 1955, pp. 61 and 62). In order to express the orientation of the simplex by the sign of this product, we make the points anticommute. Points within the simplex are expressed as formal real linear combinations of the vertex points with positive coefficients summing to unity. These points constitute the *span* of the simplex.

Thus, before forming the quantum theory, we express the classical simplex as a formal Grassman product of its vertices, and its span as the convex set of probability distributions or incoherent superpositions of its vertices.

3.3. Quantization. Since the classical theory is already antisymmetric we use Fermi quantization for the quantum theory. In the following familiar parallel only the names have been changed.

Heisenberg quantization includes:

H1. Replace $pq - qp = 0$ by $pq - qp = ih$, with fundamental phase space area h.

H2. Allow coherent as well as incoherent superposition.

Topological quantization shall consist of:

T1. Replace $pq + qp = 0$ by $pq + qp = a$, with fundamental time space element a.

T2. Allow coherent as well as incoherent superposition.

We suggest that topological quantization is the fundamental one, and that the dynamical quantization of Heisenberg is an approximate consequence.

Accordingly we replace the Grassman product defining the classical simplex by a Clifford product, and replace the incoherent superpositions defining the span by coherent. This brings us to the following concept:

3.4. The Quantum Simplex. *Definition. A quantum simplex of dimension n* is a quantum object whose Hilbert space consists of the homogeneous S numbers of degree (grade) $n+1$, their monadic factors being interpreted as describing its vertices.

It is natural to admit unlimited superposition, even of different dimensions, and then a quantum simplex is just a quantum set with a special interpretation. The emission vector (state vector) for a quantum simplex is an S number.

The classical simplex degenerates to 0, the S number meaning "nonset," when two vertices coincide, according to the rules of Grassmann algebra. According to rule $T1$. The quantum simplex loses two dimensions instead. For example a line becomes the null set 1 instead of the undefined 0. This may be understood as a cancellation of a path with its negative.

In particular a basic quantum pentahedron is associated with five anticommuting monadic S numbers. Call them a, b, c, d, e and call the signs of their squares the *signature* of the pentahedron. If the pentahedron does not have the signature $(+++++)$ or $(-----)$ then its Clifford algebra includes a Dirac algebra. To construct this algebra, imagine one vertex, say e, at the origin. Then the lines ea, eb, ec, ed, for suitably chosen e, generate a Dirac sedenion algebra, with signature $(+---)$ or $(-+++)$.

3.5. The Quantum Simplex Points in All Directions. The classical simplex s in dimension 4 has just five vertices (Figure 1). This shows in several ways:

C1. s is the product of five points.

C2. For just five points p is it true that p is a vertex (= factor) of s?

C3. The group of mappings of the points that leaves s fixed is the symmetric group on five objects (modulo irrelevant sign changes of the five objects).

A basic quantum simplex s enjoys only the first of these three properties:

Q1. s is the Clifford product of five points (by definition).

Q2. For every point p in a five-dimensional subspace of S, p is a vertex ($:=$ factor) of s.

Q3. The group of mappings of the points that leaves s fixed is the orthogonal group in five dimensions (with signature determined by the Clifford squares of any five orthogonal vertices).

Thus the classical simplex breaks orthogonal invariance (C3), while the quantum simplex generates its own orthogonal invariance (Q3).

3.6. The Quantum Complex. Classically, a complex is a set of simplexes closed under the boundary operator. So we must formulate the ideas of "set of simplexes" and "boundary" in quantum topology. The first is immediate. In quantum set theory, set and simplex are synonymous. A quantum complex is therefore a simplex whose points are simplexes, and it is a simplex whose points are sets, which means it is just a simplex. (It hardly pays to exclude the simplex 1 from the complexes.)

Thus the quantum complex too is described by an S number. But the interpretation is, perhaps, different: the factors of this number are taken to be simplexes in the complex, not points in the simplex.

The theory of the boundary operator is longer and will be presented elsewhere.

The five points of each pentad have ten bilinear products, generators of the isotropy group of the pentahedron. We have identified six of these products with Lorentz generators, for example. If two pentahedrons are not contiguous—have no points in common—then the isotropy group of one pentahedron leaves invariant the points of the other and the two isotropy groups commute with each other. This commutativity ultimately follows from the anticommutativity of distinct points, monadic S numbers, and reproduces the characteristic relation between two local groups in a theory with a gauge invariance group. Our isotropy group is thus local, not global. It is an optimistic but natural speculation that this isotropy is the raw material for all the gauge groups of Nature, including the gravitational.

6. DISCUSSION

The classical pentahedron points in five directions, the quantum one points in all. This means that the quantum simplicial model of time space may be free from a great impediment to the classical simplicial model. The classical simplicial model breaks Lorentz invariance everywhere, makes conservation of angular momentum mysterious. The quantum one preserves a Lorentz invariance for every pentahedron of suitable signature.

In the same way we can demonstrate that all quantum simplexes of the same dimension and signature are congruent. While classically each Euclidean n-simplex carries $n(n+1)/2$ length parameters as coordinates, indicating its infinite internal structure, the quantum simplex carries only its signature. If this signature is restricted to a physically suitable value, say $(--+++)$, then the resulting pentahedron is essentially unique, as befits a plexon.

We build here on an unfinished foundation. What determines the dimension and signature of the pentahedron and thus of time space? As yet the theory could accomodate any signature, and Nature prefers time-space signature $(+---)$ or $(-+++)$. We mention some speculations on this question in passing.

One way to fix the signature is to postulate a causal order among the monads of a pentahedron. Aleksandrov (1955) and Pimenov (1968) among others point out that a causal relation determines the signature of its time space to be $(+--\cdots-)$ or $(-++\cdots+)$.

When we draw a Feynman diagram, we do not expect its arrows to partially order its points. Then loops express not a causal anomaly but pair creation and annihilation.

Now, however, we are making the time space that underlies even Feynman diagrams, and which is causally ordered in all the theories of the present day. We seem to lose nothing by retaining the common assumption of a global causal order, except Shestov's principle of anarchy: "Everything is possible."

If in addition the fundamental physical quantities were the linking numbers of strings, this would fix the dimension, for it is well known that only in three spatial dimensions can strings link one another or themselves.

An alternative speculation: The ratio $n_1 : n_2$ of monads to dyads is $1:2$ only for the pentahedron. If energetic considerations favor this ratio (which recalls the virial theorem for potential and kinetic energy ratios in the Kepler problem) then the equilibrium expectation value of the simplicial dimension would be 4, and other dimensions would appear only in fluctuations from equilibrium.

Finally we point out what may be merely an irrelevant coincidence: If we generate S numbers rank by rank beginning with 1, which has rank 0, and quit when we reach a rank that includes a monadic S number of negative signature, a "timelike direction," we will find ourselves with the ranks $0, 1, 2, 3$, and a 16-dimensional subalgebra of S isomorphic to the Dirac sedenions of special relativity, with three space dimensions and one time.

We will have to build both time space coordinates x and field variables $f(x)$ out of the S numbers for the simplexes of the world. Since the anticommutation relations between remote simplexes are trivial, the first field operators to be built will not be the canonical ones but those of the third quantized, hyperquantized, or generating functional schemes.

We have used the same bracket operator Br to make complexes out of simplexes, simplexes out of points, and points out of their constituent sets, just as we use one membership relation for the analogous classical constructions in the language of set theory. This simplifies vertically, a greater unification than the unified field theory, which simplifies horizontally and on only one level. We call a theory like this, with but one basic concept, a *monophysics.*

Our hunt for the universe is still close to home, but we have encountered encouraging spoor and a section of clear trail lies ahead: the development of quantum simplicial topology.

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